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# Geometry of KDV (2): Three Examples

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Let $Q$ be a 1-dimensional Schrödinger operator with spectrum bounded from $-\infty$. By *addition* I mean a map of the form $Q \to Q' = Q - 2D^2 \lg e$ with $Qe = \lambda e$, $\lambda$ to the left of spec $Q$, and either $\int_{-\infty}^{0} e^2$ or $\int_0^\infty e^2$ finite. The *additive class* of $Q$ is obtained by composite addition and a subsequent closure; it is a substitute for the KDV invariant manifold even if the individual KDV flows have no existence. KDV(1) = McKean [1987] suggested that the additive class of $Q$ is the same as its *unimodular spectral class* defined in terms of the $2 \times 2$ spectral weight $dF$ by fixing (a) the measure class of $dF$, and (b) the value of $\sqrt{\det dF}$. The present paper verifies this for (1) the scattering case, (2) Hill's case, and (3) when the additive class is finite-dimensional (Neumann case).

**KEY WORDS**: Schrödinger operator; addition; additive class; KDV manifold; unimodular isospectral class.

## 0. INTRODUCTION

This is the second of three papers, designated KDV (1), (2), and (3). Let $Q$ be the Schrödinger operator[2] $-D^2 + q(x)$ with potential of class $C^\infty$, subject to a single condition: *that its spectrum be bounded from* $-\infty$. McKean[(9)] [KDV (1)] introduced the *additive class* and the *unimodular spectral class* of $Q$, as will now be recalled.

**Addition.** Let spec $Q$ start at 0, for definiteness, and fix $\lambda < 0$. Then $Qh = \lambda h$ has positive solutions $h_- \in L^2(-\infty, 0]$ with $\int_0^\infty h_-^2 = \infty$, and $h_+ \in L^2[0, \infty)$ with $\int_{-\infty}^0 h_+^2 = \infty$, subject to $[h_-, h_+] = 1$.[3] The point $\mathfrak{p}$ is comprised of the *projection* $\lambda$ and a *signature* $+$ or $-$. Let $e(x, \mathfrak{p})$ be $h_-(x)$

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[2] $D$ signifies differentiation by $x$.

[3] The bracket is Wronski's determinant: $[h_-, h_+] = h'_- h_+ - h_- h'_+$.

or $h_+(x)$ in accordance with the latter. Then, *addition* of $\mathfrak{p}$ to (the divisor of) $Q$ is the map

$$A^{\mathfrak{p}}: \quad Q \to Q - 2D^2 \lg e(x, \mathfrak{p})$$

These transformations commute among themselves:

$$A^{\mathfrak{p}_1}A^{\mathfrak{p}_2}: \quad Q \to Q - 2D^2 \lg[e(x, \mathfrak{p}_1), e(x, \mathfrak{p}_2)]$$

they can also be inverted: in fact,

$$A^{\mathfrak{p}}A^{-\mathfrak{p}} = 1$$

with $-\mathfrak{p}$ having the same projection as $\mathfrak{p}$ but the opposite signature. The *additive class* of $Q$ is now declared to be the smallest family of such operators $Q'$ that includes $Q$ and is closed in some technical sense to be determined by further investigation. It is a substitute for the *KDV invariant manifold* of $Q$ even if the KDV flows have no existence, as for the oscillator $Q = -D^2 + x^2 - 1$. This interpretation rests upon three facts:

1. If $\mathfrak{p} = (\lambda, +)$ and $\mathfrak{p}' = (\lambda + \Delta\lambda, +)$, then

$$A^{\mathfrak{p}'}A^{-\mathfrak{p}}: \quad Q \to Q - XQ\,\Delta\lambda - \quad \text{etc.} \qquad \text{with}^4 \quad XQ = 2G'_{xx}(\lambda)$$

so that the vector field $X$ appears as an *infinitesimal addition.*

2. $XQ$ can be developed in negative half-integral powers of $\lambda$ in the vicinity of $-\infty$:

$$XQ = \sum_1^{\infty} \lambda^{-1/2-n} X_n Q$$

in which $X_1$ = infinitesimal translation, $X_2$ = KDV, etc. are the conventional KDV fields, up to unimportant constant factors.

3. The fields $X$ can be integrated without obstruction in the additive class to produce commuting flows.

Points 1 and 2 can be found in KDV (1). The flows cited under point 3 are used in Section 4. KDV (3)[10] is devoted to them.

[4] $G_{xy}(\lambda)$ is Green's function, i.e., it effects the map $(Q-\lambda)^{-1}$. The prime signifies differentiation *on diagonal.*

**Isospectrality.** The solutions $h_-$ and $h_+$ of $Qh=\lambda h$ may be formed so as to be analytic for complex values of $\lambda$ off the *cut* $[0, \infty) \supset \operatorname{spec} Q$.[5] Now, the imaginary part $B$ of the *fundamental matrix*

$$M=A+\sqrt{-1}\,B=\begin{bmatrix} 2h_-h_+ & h'_-h_++h_-h'_+ \\ h'_-h_++h_-h'_+ & 2h'_-h'_+ \end{bmatrix} \quad \text{taken at} \quad x=0$$

is positive (-definite) in the open upper half-plane $[\lambda=a+\sqrt{-1}\,b\colon b>0]$ so that

$$B(\lambda)=\frac{b}{\pi}\int_0^\infty [(\lambda'-a)^2+b^2]^{-1}\,dF(\lambda')$$

with the positive[6] $2\times 2$ *spectral weight* $dF=[df_{ij}\colon 1\leqslant i, j\leqslant 2]$.[7] The *isospectrality* [ = unitary equivalence] of two operators $Q$ and $Q'$ is reflected in a relation[8] $dF'(\lambda)=G(\lambda)\,dF(\lambda)\,G^\dagger(\lambda)$ between their spectral weights, in which the factor $G$ takes its values in $GL(2, R)$. The narrower classification of *unimodular isospectrality* means that $G$ takes its values in $SL(2, R)$: $\det G=1$. This is the case for addition: if $Q'=A^{\mathfrak{p}_0}Q$ with the projection $\lambda(\mathfrak{p}_0)$ to the left of spec $Q$, then[(9)] $dF'=G\,dF\,G^\dagger$ with the unimodular factor

$$G(\lambda)=\frac{1}{[\lambda-\lambda(\mathfrak{p}_0)]^{1/2}}\begin{bmatrix} c & 1 \\ \lambda-\lambda(\mathfrak{p}_0)-c^2 & c \end{bmatrix}, \qquad c=\frac{e'(0,\mathfrak{p}_0)}{e(0,\mathfrak{p}_0)}$$

In particular, the additive class of $Q$ is part of its unimodular spectral class. KDV (1) put forward the conjecture that, subject to suitable technical precautions, *the additive class and the unimodular spectral are always one and the same.*

The object of the present paper is to confirm this in the three most important special cases: (1) *the scattering case*,[9] $C_\downarrow^\infty$; (2) *Hill's case*,[10] $C_1^\infty$; (3) *when the additive class is finite-dimensional.* I do not know how to proceed further. I do not even know how to prove that the general additive and/or unimodular class is a manifold, though I believe it to be so without exception.

[5] This is not really necessary, as it is only *automatically analytic* combinations such as $h_-h_+$, $-h'_+/h_+$, *etc.* that are employed, but it serves to fix ideas.

[6] The adjective means $df_{11}\geqslant 0$, $df_{22}\geqslant 0$, and $(df_{11}\,df_{22})^{1/2}\geqslant |df_{12}|$ with the natural interpretation of the radical.

[7] Weyl[(14)]; but see Kodaira[(5)] for the stylish method adopted here.

[8] The dagger signifies transpose.

[9] $C_\downarrow^\infty$ is the class of infinitely differentiable functions vanishing rapidly at $\pm\infty$.

[10] $C_1^\infty$ is the class of infinitely differentiable functions of period 1.

## 1. SCATTERING CASE

$Q$ is taken without bound states, for simplicity, but see Section 3 for how they may be included.

**Standardization at $\infty$.** I review the conventional scattering theory of Faddeev[(2)] and Marcenko.[(6),11] Let $k=\sqrt{\lambda}\neq 0$ be real, positive values of $k$ corresponding to the upper bank of the cut spec $Q=[0,\infty)$ and negative values to the lower bank. At $x=\pm\infty$, any solution of $Qf=k^2f$ is a combination of the free solutions $\exp(-\sqrt{-1}\,kx)$ and $\exp(\sqrt{-1}\,kx)$, or nearly so, and this fact may be used to single out two standard solutions $f_-$ and $f_+$ with the comportment at $\pm\infty$ indicated in Table I: $f_+$ has the form of a wave $\exp(\sqrt{-1}\,kx)$ entering from $-\infty$, one part $s_{11}(k)\exp(\sqrt{-1}\,kx)$ being *transmitted* to $+\infty$ and the other $s_{12}(k)\exp(-\sqrt{-1}\,kx)$ being *reflected* back to $-\infty$. The *scattering matrix*

$$[s_{ij}(k)\colon 1\leqslant i,j\leqslant 2]=[s_{ij}^*(-k)]$$

of *transmission* and *reflection coefficients* is fully specified by the recipe: it is of class $C^\infty[R, SU(2)]$ after extension to $k=0$ and tends rapidly to the identity at $\pm\infty$; in particular, $|s_{11}|^2+|s_{12}|^2=1$, $s_{12}=-s_{21}^*s_{11}/s_{11}^*$, and $s_{21}\in C_\downarrow^\infty$. One finds $[f_-,f_+]$ to be $-2\sqrt{-1}\,k\,s_{11}$ by evaluation at $+\infty$, and comparison at $-\infty$ reveals that $s_{11}=s_{22}$. The independence of $f_-$ and $f_+$ for $k\neq 0$ means that $s_{11}$ does not vanish in that case; $s_{11}(0)$ *can* vanish, in which case[12] $\dot{s}_{11}(0)\neq 0$. The scattering matrix can be viewed as providing a *patching* across the cut of the functions $f_-(x)$ and $f_+(x)$ attached to the upper bank $(k>0)$ and their conjugates $f_-^*(x)$ and $f_+^*(x)$ attached to the lower bank $(k<0)$:

$$f_+^*(x)=s_{11}^*f_-(x)+s_{12}^*f_+(x)$$
$$f_-^*(x)=s_{21}^*f_-(x)+s_{22}^*f_+(x)$$

[11] Deift and Trubowitz[(1)] is recommended as the most careful treatment.
[12] The dot signifies differentiation by $k$.

**Table I.**

| | $x\downarrow -\infty$ | $x\uparrow +\infty$ |
|---|---|---|
| $f_+(x)$ | $e^{\sqrt{-1}\,kx}+s_{12}(k)\,e^{-\sqrt{-1}\,kx}$ | $s_{11}(k)\,e^{\sqrt{-1}\,kx}$ |
| $f_-(x)$ | $s_{22}(k)\,e^{-\sqrt{-1}\,kx}$ | $e^{-\sqrt{-1}\,kx}+s_{21}(k)\,e^{\sqrt{-1}\,kx}$ |

The transmission coefficient $s_{11}$ extends to the open upper half-plane as an *outer function*:

$$s_{11}(k) = \exp\left[\frac{1}{\pi\sqrt{-1}}\int_{-\infty}^{\infty}(k'-k)^{-1}\lg|s_{11}(k')|\,dk'\right]$$

In particular, $s_{21}$ determines $s_{11}$, and so also the whole scattering matrix, via this recipe and the identity $|s_{11}|^2 = 1 - |s_{21}|^2$; moreover, $s_{11}$ is of class $H^\infty$, being of modulus $\leqslant 1$, and of class $1 + H^2$ in view of the rapid vanishing of $s_{21}$.

**Backward Scattering.** The same is true of the auxiliary functions

$$e_+(x) = \exp(-\sqrt{-1}\,kx)\,f_+(x)/s_{11}$$
$$e_-(x) = \exp(\sqrt{-1}\,kx)\,f_-(x)/s_{11}$$

They are outer functions of class $H^\infty$ and also of class $1 + H^2$. Now the patching takes the form

$$e_+^*(x) + s_{21}\exp(2\sqrt{-1}\,kx)\,e_+(x) = s_{22}e_-(x)$$
$$e_-^*(x) + s_{12}\exp(-2\sqrt{-1}\,kx)\,e_-(x) = s_{11}e_+(x)$$

leading to a very beautiful proof[1] that $s_{21}$ determines $Q$. The $Q$ and $Q'$ have the same reflection coefficient $s_{21}$ only if they have common transmission coefficient $s_{11}$ as well. Then the differences $\Delta e_-$ and $\Delta e_+$ are of class $H^2$,

$$\Delta e_+^* + s_{21}\exp(2\sqrt{-1}\,kx)\,\Delta e_+ = s_{11}\,\Delta e_- \in H^2$$

and the integral over the whole line of the function

$$|\Delta e_+|^2 + s_{21}\exp(\sqrt{-1}\,kx)(\Delta e_+)^2 = s_{11}\,\Delta e_-\,\Delta e_+ \in H^1$$

must vanish. But this cannot be maintained in the face of $|s_{21}| < 1$ $(k \neq 0)$ unless $\Delta e_+ = 0$, i.e., $Q = Q'$. This may be rephrased: *for fixed $s_{11}$, the phase of $s_{21}$ determines $Q$.* Besides, the conditions

1. $s_{21} \in C_\downarrow^\infty$
2. $s_{21}^*(-k) = s_{21}(k)$
3. $|s_{21}(k)| < 1$ $(k \neq 0)$
4. $s_{21}(0) = -1$ if $|s_{21}(0)| = 1$

are not only necessary, but also sufficient that $s_{21}$ be the reflection coefficient of an operator $Q$ of scattering class, so that the family of operators with fixed transmission coefficient is *either* the singleton $Q = -D^2$ with transmission identically 1, *or else* it is in natural correspondence with odd functions phase $s_{21}$ of class $C^\infty$.

**Additive Class.** KDV (1) identified the additive class of $Q$ as the preceding family with common transmission coefficient: in fact, addition of $\mathfrak{p}_0 = (-k_0^2, \pm)$ multiplies $s_{21}$ by[13] $(k_0 + \sqrt{-1}\,k)[(k_0 - \sqrt{-1}\,k)]^{-1}$ if the signature is positive and by its reciprocal in the opposite case, so that *subtraction* of $(-k_0^2, +)$ followed by *addition* of $(-(k_0 + 1/m)^2, +)$, repeated $n$-fold, results in the addition to phase $s_{21}$ of

$$n \times \text{phase} \frac{k_0 - \sqrt{-1}\,k}{k_0 + \sqrt{-1}\,k} \frac{k_0 + 1/m + \sqrt{-1}\,k}{k_0 + 1/m - \sqrt{-1}\,k} = \left[\frac{n}{m} + o(1)\right] \times \frac{-2k}{k_0^2 - k^2}$$

if $n \uparrow \infty$ and $m \to \pm\infty$ at comparable speed, from which it appears that *composite addition and a careful passing to the limit can add to* phase $s_{21}$ *any odd function of class* $C^\infty$ *one pleases.*

**Unimodular Class.** The $2 \times 2$ spectral weight of $Q$ has no singular part, its density relative to $d\lambda = 2k\,dk$ being[14]

$$F'(\lambda) = B = \operatorname{Im} \frac{1}{-2\sqrt{-1}\,k\,s_{11}} \begin{bmatrix} 2f_- f_+ & f'_- f_+ + f_- f'_+ \\ & 2f'_- f'_+ \end{bmatrix} \quad \text{at} \quad x = 0$$

$$= \frac{|s_{11}|^2}{2k} \begin{bmatrix} |e_-|^2 + |e_+|^2 & \operatorname{Re}(e_-^* e'_- + e_+^* e'_+) \\ & |e'_-|^2 + |e'_+|^2 \end{bmatrix} \quad \text{at} \quad x = 0$$

Now $sp\, dF = df_{11} + df_{22}$ typifies the Lebesgue measure class on spec $Q = [0, \infty)$ and $(\det dF)^{1/2} = |s_{11}|\, d\lambda$,[(9)] and it is the *precise value* and not just the *measure class* of the latter that specifies the *unimodular class* of $Q$.

**Standardization at 0.** To identify the unimodular class of $Q$ with its additive class (= the KDV invariant manifold), it will be necessary to deal with operators $Q'$ not yet known to be of scattering class and so not yet amenable to standardization at $\infty$. The only *general* alternative is to standardize at $x = 0$. I explain how to do this for $Q$ of scattering class. The solutions $h_- \in L^2(-\infty, 0]$ and $h_+ \in L^2[0, \infty)$ of $Qh = \lambda h$ are partially standardized by $[h_-, h_+] = 1$. They are taken positive for $\lambda < 0$ (imaginary $k$) and may be formed so as to extend analytically off the cut $[0, \infty) = \text{spec}\, Q$, and since they do not vanish there, it is permitted also to *standardize at* 0 by taking $h_-(0) = h_+(0)$. Then $f_+ = ch_+$ with a nonvanishing factor $c$, $f_- = -2\sqrt{-1}\,ks_{11}h_-/c$ in view of $[f_-, f_+] = -2\sqrt{-1}\,ks_{11}$, and

[13] $k_0$ is reckoned positive.

[14] I omit the lower left entry by reason of symmetry here and below.

$-c^2/2\sqrt{-1}\,k = s_{11} f_+(0)/f_-(0)$ in view of $h_-(0) = h_+(0)$. Now, $h_-$ and $h_+$ extend smoothly to the banks of the cut and can be patched across it:

$$h_+^*(x) = r_{11} h_-(x) + r_{12} h_+(x)$$
$$h_-^*(x) = r_{21} h_-(x) + r_{22} h_+(x)$$

with

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} -2\sqrt{-1}\,k\,|s_{11}/c|^2 & s_{12}^* c/c^* \\ s_{12} c^*/c & |c|^2/2\sqrt{-1}\,k \end{bmatrix}$$

$-c^2/2\sqrt{-1}\,k = s_{11} e_+(0)/e_-(0)$ being the outer function in the half-plane $[k = a + \sqrt{-1}\,b\colon b > 0]$ with modulus $|r_{22}|$ on the bordering line:

$$\frac{-c^2}{2\sqrt{-1}\,k} = \exp\left[\frac{1}{\pi\sqrt{-1}} \int_{-\infty}^{\infty} (k' - k)^{-1} \lg |r_{22}(k')|\, dk'\right]$$

**Preview.** The plan of attack on the operators $Q'$ of the unimodular class is as follows: (a) introduce the patching coefficients $[r_{ij}\colon 1 \leqslant i, j \leqslant 2]$ as for the scattering class; (b) check that $|r_{22}|$ is the modulus of an outer function $-c^2/2\sqrt{-1}\,k$; (c) define $s_{12}$ as $r_{21} c/c^*$ and verify that this is consistent with $|s_{11}| = |c|(|r_{11}|/2\,|k|)^{1/2}$; (d) verify that $s_{12}$ is the reflection coefficient of an operator $Q$ of scattering class; (e) verify that $[r_{ij}]$ determines $Q'$; (f) check that $s_{12}$ determines $c$ and so also the whole of $[r_{ij}]$ so that the map $Q' \to s_{12}$ is 1:1 and $Q' = Q$. The plan requires three *technical conditions* besides the two (nontechnical) conditions that determine the unimodular class, *to wit*:

1. sp *dF typifies the Lebesgue measure class on* $[0, \infty)$
2. $\det B = |s_{11}|^2$

**Technical Condition #1.** $B = [b_{ij}\colon 1 \leqslant i, j \leqslant 2]$ is smooth on the bordering line except perhaps at $k = 0$.

**Technical Condition #2.** Near $k = 0$, $B$ is of the form $kC$ or $C/k$ according as $s_{11}(0)$ vanishes or not, $C = [c_{ij}]$ being smooth and $c_{11}(0) \neq 0$.

**Technical Condition #3.** $\int_0^\infty (kb_{11} - 1)^2\, dk < \infty$.

**Discussion.** In the scattering case, conditions #1 and #2 are self-evident from the form of $B$ and the fact that $e_-(0)$ and $e_+(0)$ are smooth on the (punctured) bordering line. Condition #3 follows from the fact that

$$2kb_{11} = |s_{11}|^2[|e_-(0)|^2 + |e_+(0)|^2]$$

is smooth at $k=0$ and from the comportment of $s_{11}e_{\pm}(0)$ at $k=\infty$: both these functions are bounded and of class $1+L^2[0,\infty)$.

**Identification of the Unimodular and the Additive Classes.** The operator $Q'$ is fixed, subject to the unimodular conditions 1 and 2 and to the technical conditions #1, #2, and #3. It is to be proved *that $Q'$ itself is of scattering class.*

**Step 1.** The fundamental matrix $M=A+\sqrt{-1}\,B$ is formed and it is noted, from technical condition #1, that $B$ is smooth in the closed half-plane punctured at $k=0$. It is an elementary consequence of the Cauchy–Riemann equations that $A$ shares this feature, so that $m_{11}=2h_+^2(0)$, $m_{12}=[h'_-(0)+h'_+(0)]\,h_+(0)$, and $[h'_-(0)-h'_+(0)]\,h_+(0)$ $(=1)$ are also smooth down to the punctured line. Besides, $h_+(0)$ does not vanish there, since $b_{11}=2\,\mathrm{Im}\,h_+^2(0)=0$ entails $\det B=b_{11}b_{22}-b_{12}^2=|s_{11}|^2=0$ and $s_{11}$ cannot vanish unless $k=0$. The upshot is that $h_-(0)=h_+(0)$, $h'_-(0)$, and $h'_+(0)$ are smooth down to the punctured line. This was the aim of Step 1.

**Step 2** is to introduce the patching coefficients $[r_{ij}]$ as before:

$$h_+^*(x)=r_{11}h_-(x)+r_{12}h_+(x)$$
$$h_-^*(x)=r_{21}h_-(x)+r_{22}h_+(x)$$

These are unambiguous and also smooth on the punctured line in view of

$$r_{11}=[h_+^*,h_+],\qquad r_{12}=[h_-,h_+^*]$$
$$r_{21}=[h_-^*,h_+],\qquad r_{22}=[h_-,h_-^*]$$

Besides, from the fact that $**=$ *the identity* it follows that $-r_{11}r_{22}+|r_{12}|^2=1$ upon noting that $r_{12}^*=r_{21}$ and that $r_{11}$ and $r_{22}$ are both imaginary: in fact,[(9)]

$$|s_{11}|^2=\det B=1-|[h_-,h_+^*]|^2=1-|r_{12}|^2=-r_{11}r_{22}$$

so that $r_{11}$ and $r_{22}$ have the same (imaginary) signature and $|r_{12}|^2=1-|s_{11}|^2$ is of class $C_\downarrow^\infty$.

**Step 3** introduces $s_{12}$. The first point to be verified is that $|r_{22}|$ is the modulus of an outer function: indeed, $r_{22}=h_-^*(0)/h_-(0)-r_{21}$ in view of $h_-(0)=h_+(0)$, so $|r_{22}|\leqslant 1+|r_{21}|\leqslant 2$ and $|r_{22}|\geqslant\frac{1}{2}(1-|r_{21}|^2)=\frac{1}{2}|s_{11}|^2$, whence[15]

$$\int_{-\infty}^{\infty}(1+k^2)^{-1}\lg^+|r_{22}|\,dk\leqslant\pi\lg 2$$

[15] $\lg^+$ and $\lg^-$ are the positive and negative parts of the logarithm, respectively.

and

$$\int_{-\infty}^{\infty} (1+k^2)^{-1} \lg^- |r_{22}|\, dk$$

$$\leqslant \pi \lg 2 + 2\int_{-\infty}^{\infty} (1+k^2)^{-1} \lg^- |s_{11}|\, dk$$

are both finite. The outer function $-c^2/2\sqrt{-1}\,k$ with modulus $|r_{22}|$ may now be formed and the reality condition $r_{22}^*(-k) = r_{22}(k)$ may be used to verify that the outer function $c^2$, and so also its root $c$, satisfies the reality condition. Now, $r_{22}$ does not vanish on the punctured line, since $-r_{11}r_{22} = |s_{11}|^2$, so $\lg |r_{22}|$ is smooth there and $c$ inherits both properties: smoothness and no roots. This allows one to write

$$\begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix} = \begin{bmatrix} -2\sqrt{-1}\,k\,|s_{11}/c|^2 & s_{12}^* c/c^* \\ s_{12}c^*/c & |c|^2/2\sqrt{-1}\,k \end{bmatrix} \qquad \text{if} \quad k \neq 0$$

with $s_{12}$ subject to the reality condition and $|s_{12}|^2 + |s_{11}|^2 = 1$, *provided* $r_{11}$, and so also $r_{22}$, is *negative* imaginary on the upper bank of the cut ($k > 0$). This follows from an examination of $r_{11}$ extended to the sector $[k = a + \sqrt{-1}\,b\colon\ a, b \geqslant 0]$ by means of $r_{11} = [h_+^*, h_+](0)$: in fact, the extended function is imaginary and root-free in the open sector, while, on the half-line $a = 0$, it vanishes, $h_+(x)$ being real, and[16]

$$\partial r_{11}/\partial a = 2[h_+, \dot{h}_+] = -4\sqrt{-1}\,b \int_0^{\infty} h_+^2(x)\, dx$$

is negative imaginary. This does the trick.

**Step 4.** The function $s_{12}$ is smooth on the punctured line, $s_{12}^*(-k) = s_{12}(k)$, and $|s_{12}|^2 = 1 - |s_{11}|^2$, so that $s_{12}$ vanishes rapidly at $\pm\infty$. Now in order that $s_{12}$ be a realistic reflection coefficient determining an operator $Q$ of scattering class, it is necessary still to verify that *it is smooth at* $k = 0$ *and takes the value* $-1$ *if* $s_{11}(0)$ vanishes. This is what the technical condition #2 is for.

*Case 1.* $s_{11}(0) = 0$. $B = kC$, so the fundamental matrix $M$ may be expressed near $k = 0$ as

$$M(k^2) = (k/\pi) \int_{-1}^{1} (k' - k)^{-1}\, C(k')\, dk'$$

[16] The dot signifies differentiation by $a$. The formula is standard.

up to a smooth additive correction. It follows that $M$ is smooth in the vicinity of $k=0$, and since $0<m_{11}(k^2)$ increases for $k\downarrow\sqrt{-1}\,0+$, then $m_{11}=2h_+^2(0)$ does not vanish at $k=0$. Then $h_+(0)=(m_{11}/2)^{1/2}$, $h'_-(0)-h'_+(0)=m_{12}/h_+(0)$, and $h'_-(0)-h'_+(0)=1/h_+(0)$, and so also $h'_-(0)$ and $h'_+(0)$ individually are smooth at $k=0$, and $[r_{ij}]$ inherits this feature. Now $-r_{11}r_{22}=|s_{11}|^2$ imitates a multiple of $k^2$ and $r_{22}=[h_-^*, h_-]$ changes sign at $k=0$. This can happen only if $r_{22}(k)\approx r^{\cdot}_{22}(0)\,k$, with $r^{\cdot}_{22}(0)\neq 0$: the only other possibilities ($r_{22}$ vanishing twice or not at all) do not permit the signature change. But then $|2kr_{22}|$, alias the modulus of the outer function $c^2$, imitates $2\,|r^{\cdot}_{22}(0)|\,k^2$ in the vicinity of $k=0$, and $c^2$ itself imitates a *negative*[17] multiple of $k^2$, so that $c/c^*$ is smooth, and with it also $s_{12}=r_{21}c/c^*$. The value $s_{12}(0)=-1$ is easily elicited: $r_{11}$ vanishes in the same style as $r_{22}$ and $r_{11}+r_{12}=r_{21}+r_{22}$, so $r_{12}(0)=r_{21}(0)$ is real and of modulus $[1+r_{11}(0)\,r_{22}(0)]^{1/2}=1$, and since $c/c^*=-1$ at $k=0\pm$, it suffices to rule out $r_{21}(0)=-1$ by means of the implication $h_-^*(0)=-h_-(0)$, which contradicts the positivity of $m_{11}=2h_-^2(0)$ at $k=0$.

*Case 2.* $s_{11}(0)\neq 0$. The discussion is similar. Now $B=C/k$, $k[r_{ij}]$ is smooth, $r_{11}\approx k$, and $r_{22}\approx 1/k$ up to constant multipliers (or the other way around), $c^2$ imitates a *positive* constant in the vicinity of $k=0$, and $c/c^*$ is smooth, as is $r_{21}$ and so also $s_{12}$. The details are left to the reader.

**Step 5.** It is to be proved that $Q'$ is one and the same as the scattering class operator $Q$ determined by the reflection coefficient $s_{12}$. The present step elicits the preliminary fact that $[r_{ij}]$ determines $Q'$. The point of departure is the fact that the harmonic function $\psi=\operatorname{imag}\lg m_{11}(k^2)$ is bounded between 0 and $\pi$ in the open sector $[k=a+\sqrt{-1}\,b\colon a, b>0]$ in view of the positivity of $\operatorname{Im} m_{11}$ for $\operatorname{Im} k^2>0$. Now[18]

$$r_{11}+r_{22}=-2\sqrt{-1}\sin\psi \text{ and } r_{12}+r_{21}=2\cos\psi$$

determines this function on the boundary of the sector and so in the large. Then $m_{11}$ itself is known up to a multiplicative factor, which is fixed by the development $m_{11}(\lambda)\approx(-\lambda)^{-1/2}$ at $-\infty$, and so one also knows

$$r_{11}\,|h_+(0)|^{-2}=2\operatorname{Im} h'_+(0)/h_+(0)$$

which determines $Q'$ on the half-line $x\geqslant 0$ (Ref. 3; see also Ref. 9); similarly,

$$r_{22}\,|h_-(0)|^{-2}=-2\operatorname{Im} h'_-(0)/h_-(0)$$

determines it on the left half-line.

[17] $c^2$ is *positive* for imaginary values of $k$.
[18] $h'_-(0)-h'_+(0)=1/h_+(0)$ is used; also, $m_{11}=2h_+^2(0)$.

**Step 6.** The proof is finished by checking that $s_{12}$, or, what is the same, $s_{21} = -s_{12}^* s_{11}/s_{11}^*$, determines the outer function $c$ and so also the whole of $[r_{ij}]$. To do this, I introduce the functions $e_+(0) = ch_+(0)/s_{11}$ and $e_-(0) = -2\sqrt{-1}\,kh_-(0)/c$ and note the *patching*:

$$e_+^*(0) + s_{21}e_+(0) = s_{11}e_-(0)$$
$$e_-^*(0) + s_{12}e_-(0) = s_{11}e_+(0)$$

It follows that $e_+(0)$ and $e_-(0)$, and so also $-c^2/2\sqrt{-1}\,k = s_{11}e_+(0)/e_-(0)$, are determined by $s_{21}$, as in the section on backward scattering, *provided* $e_-(0)$ and $e_+(0)$ are of class $1 + H^2$: in fact, it suffices to deal with $e_+(0)$, since $e_-(0)$ is changed into $e_+(0)$ by the reflection $x \to -x$. The proof is subdivided into brief items.

**Item 1.** $-c^2/2\sqrt{-1}\,k$ is of class $H^\infty$.

*Proof.* This function is outer and

$$\left| -\frac{c^2}{2\sqrt{-1}\,k} - s_{12} \right| = \left| \frac{e_+(0)}{e_-(0)} s_{11} - s_{12} \right| = \left| \frac{e_-^*(0)}{e_-(0)} \right| = 1$$

by the second line of the patching recipe of Step 6, so $-c^2/2\sqrt{-1}\,k$ is of modulus not more than 2 on the bordering line, and this bound is inherited in the half-plane.

**Item 2.** $-c^2/2\sqrt{-1}\,k$ is of class $1 + H^2$.

*Proof.* Let $\theta$ be the phase of $-c^2/2\sqrt{-1}\,k$: it is the conjugate function (Hilbert transform) of $\lg |r_{22}|$, and since this function is of class $L^2$, we also have $\int \theta^2 < \infty$; in particular,[19]

$$\begin{aligned} |-c^2/2\sqrt{-1}\,k - 1| &= \big||r_{22}| \exp(\sqrt{-1}\,\theta) - 1\big| \\ &= \big|(|r_{22}| - 1)\exp(\sqrt{-1}\,\theta) + \exp(\sqrt{-1}\,\theta) - 1\big| \\ &\leqslant |s_{12}| + |\theta| \end{aligned}$$

so

$$\int |-c^2/2\sqrt{-1}\,k - 1|^2 \leqslant 2\int |s_{12}|^2 + 2\int \theta^2 < \infty \text{ on the border}$$

and this bound is inherited on horizontal lines in the half-plane, much as in Item 1.

[19] $1 - |s_{12}| \leqslant |r_{22}| \leqslant 1 + |s_{12}|$, so $\lg |r_{22}|$ vanishes rapidly, etc.

**Item 3.** The root $c(-2\sqrt{-1}\,k)^{-1/2}$ is of the same type: it is of class $H^\infty \cap 1 + H^2$.

*Proof.* Routine.

**Item 4.** $1/s_{11}$ is bounded on $[1, \infty)$ and of class $L^2[1, \infty)$.

*Proof.* Self-evident.

**Item 5.** $(-2\sqrt{-1}\,k)^{1/2}\,h_+(0)$, and so also $e_+(0) = ch_+(0)/s_{11}$, is an outer function.

*Proof.* It suffices to check that $4kh_+^2(0) = 2km_{11}$ is outer. Now[20]

$$\psi(k) = -2km_{11}(k^2) = -\frac{2k}{\pi}\int_0^\infty (\lambda' - \lambda)^{-1}\, df_{11} = \frac{1}{\pi}\int_{-\infty}^\infty (k' - k)^{-1}\, df_{11}$$

so[21]

$$\begin{aligned}\phi(k) &= (k - \sqrt{-1})^{-2}[\psi(k) - \psi(\sqrt{-1}) - (k - \sqrt{-1})\,\dot\psi(\sqrt{-1})] \\ &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{1}{k' - k}\, df_0\end{aligned}$$

in which $df_0 = [(k')^2 + 1]^{-1}\, df_{11}$ is of total mass $m_0 < \infty$. This permits the application of Carleson's inequality[(4)]: if $b > 0$ is fixed and if $K \subset R$ is the set of points where $|\phi(a + \sqrt{-1}\,b)| \geqslant 2^n$, then

$$\int_K (a^2 + 1)^{-1}\, da \leqslant 2^{-n} \times 4\pi m_0$$

so that

$$\int |\sqrt{\phi}|(a + \sqrt{-1}\,b)(a^2 + 1)^{-1}\, da \leqslant \pi + 4\pi m_0(\sqrt{2} - 1)^{-1}$$

independently of $b > 0$. The rest is routine: $(k + \sqrt{-1})^{-4}\,\phi$, and so also $(k + \sqrt{-1})^{-6}\,\psi$, is of class $H^{1/2}$; in particular, the latter is the product of an inner and an outer function, and since it is smooth down to the bordering line and root-free above and below, the inner part is absent.

**Item 6.** $e_+(0) - 1$ is now seen to be the product of an inner and an outer function, and to check that it is of class $H^2$ it suffices to confirm

[20] $df_{11}$ is extended from $k > 0$ to $k < 0$ symmetrically.
[21] The dot signifies differentiation by $k$.

$\int_0^\infty |e_+(0)-1|^2<\infty$; in fact, by Items 1–5, it suffices to check that $I_0=\int_0^1 |e_+^2(0)|<\infty$ and that

$$I_1=\int_1^\infty |(-2\sqrt{-1}\,k)^{1/2}\,h_+(0)-1|^2<\infty$$

*Proof That $I_0<\infty$.* $r_{11}$ and $r_{22}$ have common (imaginary) signature and

$$r_{11}+r_{22}=h_+^*(0)/h_+(0)-h_+(0)/h_+^*(0)$$

so

$$\begin{aligned}|e_+^2(0)|&=|-2\sqrt{-1}\,k|\;|-c^2/2\sqrt{-1}\,k|\;|h_+^2(0)|\;|s_{11}^{-2}|\\&=2k\,|r_{22}|\;|h_+^2(0)|\;|s_{11}^{-2}|\\&\leqslant 2k\,|h_+^2(0)^*-h_+^2(0)|\;|s_{11}^{-2}|\\&=2kb_{11}\,|s_{11}^{-2}|\end{aligned}$$

is summable on $[0,1]$ in view of technical condition #2.

*Proof That $I_1<\infty$.* It suffices to prove that $-2\sqrt{-1}\,kh_+^2(0)$ is of class $1+L^2[1,\infty)$ since, with $-2\sqrt{-1}\,kh_+^2(0)=1+\psi$,

$$|(-2\sqrt{-1}\,k)^{1/2}\,h_+(0)-1|=|(1+\psi)^{1/2}-1|$$

is majorized by, for instance, $5\,|\psi|$. But

$$-2\sqrt{-1}\,kh_+^2(0)=-\sqrt{-1}\,km_{11}=-\sqrt{-1}\,ka_{11}+kb_{11}$$

and $\int_1^\infty(kb_{11}-1)^2<\infty$ by technical assumption #3, so only $\int_1^\infty(ka_{11})^2<\infty$ is moot, and that follows from the identity[22]

$$ka_{11}+\sqrt{-1}\,(kb_{11}-1)=\frac{1}{\pi}\int_{-\infty}^{\infty}(k'-k)^{-1}(k'b_{11}-1)\,dk'$$

showing that $-ka_{11}$ is the conjugate function of $kb_{11}-1$. The proof is finished.

[22] $b_{11}$ is taken as an odd function of $k'$.

## 2. HILL'S CASE

$Q$ is now to be of period 1: its spectrum consists of (dark) *bands* $[\lambda_{n-1}^+, \lambda_n^-]$ with intervening (light) *gaps* $[\lambda_n^-, \lambda_n^+]$, as in Fig. 1, in which

$$-\infty < \lambda_0^+ < \lambda_1^- \leqslant \lambda_1^+ < \lambda_2^- \leqslant \lambda_2^+ < \cdots \uparrow +\infty$$

and the numbers $\lambda_n^-$ and $\lambda_n^+$ have the common development $n^2\pi^2 + c_0 + c_1 n^{-2} + c_2 n^{-4} + \cdots$ as $n \uparrow \infty$. I suppose, for definiteness, that all the gaps are open, but see Section 4 for the case of $g < \infty$ open gaps. The $2 \times 2$ spectral weight of $Q$ is $dF = B\, d\lambda$, in which[9]

$$B = \frac{\pm 1/2}{\sqrt{1-\Delta^2}} \begin{vmatrix} 2h_2(1) & h_2'(1) - h_1(1) \\ & -2h_1'(1) \end{vmatrix}$$

$h_1$ and $h_2$ are the solutions of $Qh = \lambda h$ with $h_1(0) = h_2'(0) = 1$ and $h_1'(0) = h_2(0) = 0$, $\Delta$ is the so-called *discriminant* $\frac{1}{2}[h_1(1) + h_2'(1)]$, and the signs alternate on the bands of spec $Q$ starting with *plus*; in particular, on spec $Q$, (1) *sp dF* typifies the Lebesgue measure class, and (2) det $B = 1$. It is to be proved that *if $Q'$ belongs to the same unimodular spectral class as $Q$, i.e.*, if its $2 \times 2$ spectral weight also satisfies conditions 1 and 2 on spec $Q$ and vanishes elsewhere, *then $Q'$ is also a Hill operator and so belongs to the additive class of $Q$.*[23] Unlike the scattering case, here no technical conditions are needed, due to an automatic compactness.

**Step 1.** Let $Q_-^0$ and $Q_+^0$ be the *side operators* obtained by restricting $Q$ to functions that live on $(-\infty, 0]$, or on $[0, \infty)$, and vanish at $x = 0$. The corresponding spectral weights are[24]

$$df_-^0 = \lim_{b \downarrow 0} -\operatorname{Im} \frac{h_-'(0)}{h_-(0)}\, da, \qquad df_+^0 = \lim_{b \downarrow 0} \operatorname{Im} \frac{h_+'(0)}{h_+(0)}\, da$$

and it is the content of Step 1 that *$df_-^0 = df_+^0$ in the open bands.* The proof can be repeated verbatim from Item 7, Section 4 of KDV (1). The moral is that the mass distribution $df_+^0 - df_-^0$ representing $\operatorname{Im}[h_+'(0)/h_+(0) + h_-'(0)/h_-(0)]$ *is concentrated in the closed gaps.*

[23] KDV (1); see also McKean and Trubowitz[12] for general background.
[24] $h'(0)/h(0)$ is taken at $\lambda = a + \sqrt{-1}\, b$ in the upper half-plane; see KDV (1).

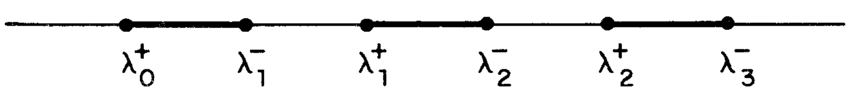


Figure 1

**Step 2.** $m_{11}=2h_-(0)\,h_+(0)$ is real, analytic, and (strictly) increasing in the open gaps,[25] and so has *at most one* (*simple*) *root in any one of them.* The functions $h_-$ and $h_+$ are determined individually at any point of an open gap as the solutions of $Qh=\lambda h$ with $h_-\in L^2(-\infty,0]$, $h_+\in L^2[0,\infty)$, and $[h_-,h_+]=1$; no further standardization is made. Then, if $h_-(0)$ or $h_+(0)$ vanishes, the other does not, and a *signature* can be ascribed to the root according as $h_-(0)=0$ or $h_+(0)=0$: in short, $df^0_+-df^0_-$ has at most one singleton mass in any *open* gap, and either it belongs to $df^0_-$ and not to $df^0_+$ or the other way around, in accordance with the signature; in particular,

$$j(0)=-[h'_+(0)/h_+(0)+h'_-(0)/h_-(0)]$$

*is a meromorphic function having* $\leqslant 3$ *poles per closed gap.*

**Step 3.** The presence of a pole of $j(0)$ *inside* a gap precludes any poles at the *ends* of the gap.

*Proof.* Let the gap be $[\lambda_1^-,\lambda_1^+]$ for definiteness, and let $r$ be the residue of $j(0)$ at the left end. A small displacement of the origin from $x=0$ to $x=x'$ changes $m_{11}$ into $2h_-(x')\,h_+(x')$ and causes the interior root (if present) to move slightly: clearly, no other interior root can appear, so the mass $r$ sticks at the left, only now it depends upon the displacement: $r=r(x')$. Let

$$j(x)=-[h'_+(x)/h_+(x)+h'_-(x)/h_-(x)]$$

Then $j(x')\approx(\lambda-\lambda_1^-)^{-1}\,r(x')$ for $\lambda$ above $\lambda_1^-$ and not too close to the next pole of this function, and from

$$j'(x)=O(1)+(h'_+/h_+)^2+(h'_-/h_-)^2\geqslant O(1)+\tfrac{1}{2}j^2(x)$$

it develops that, to leading order,

$$\begin{aligned}\frac{r(b)-r(a)}{\lambda-\lambda_1^-}&=\int_a^b j'(x)\,dx\\&\geqslant\frac{1}{2}\int_a^b j^2(x)\,dx\\&=\frac{1}{2}\int_a^b\left[\frac{r(x)}{\lambda-\lambda_1^-}\right]^2dx\end{aligned}$$

[25] $m_{11}(\lambda)=(1/\pi)\int_0^\infty(\lambda'-\lambda)^{-1}\,df_{11}$ off the spec $Q$.

for small $a \leqslant 0 \leqslant b$. This cannot be maintained for $\lambda \downarrow \lambda_1^-$ unless $r(x)=0$ for almost every value of $x$ between $a$ and $b$, in which case $r(b) \geqslant r(a)$ and $r(0)=0$ follows from $r(b) \geqslant r(0) \geqslant r(a)$ by choice of $a$ and $b$. The argument proves still more: if no interior pole is present, then a small displacement must move any end pole a little distance inside the gap, the upshot being that $j(0)$ *has at most one pole in the closed gap.*

**Step 4** is to check that *a pole of* $j(0)$ *is present in every gap.*

*Proof.* The spectral representation

$$2h_-(x)\,h_+(x) = \frac{1}{\pi}\int_{\text{bands}} (\lambda' - \lambda)^{-1}[h_1^2(x)\,b_{11} + 2h_1(x)\,h_2(x)\,b_{12} + h_2^2(x)\,b_{22}]\,d\lambda'$$

will be used to prove that $2h_-(x)\,h_+(x)$ tends to $-\infty$ as $\lambda \downarrow \lambda_n^-$ and to $+\infty$ as $\lambda \uparrow \lambda_n^+$, forcing the presence of an interior root. Let $2h_-(x)\,h_+(x)$ stay bounded as $\lambda \uparrow \lambda_n^+$, say. Then

$$\int_{\lambda_n^+}^{\lambda_{n+1}^-} (\lambda' - \lambda_n^+)^{-1}[h_1^2(x)\,b_{11} + 2h_1(x)\,h_2(x)\,b_{12} + h_2^2(x)\,b_{22}]\,d\lambda' < \infty$$

so that, by a self-evident manipulation,

$$(\lambda' - \lambda_n^+)^{-1}[h_1(x)\,b_{11}^{1/2} \pm h_2(x)\,b_{22}^{1/2}]^2 + 2\,|h_1(x)\,h_2(x)|\,[(b_{11}b_{22})^{1/2} \pm b_{12}]$$

is summable just to the right of $\lambda_n^+$. Now $[h_1, h_2]=1$, so for any choice of $x=x_1$, one can find a nearby point $x=x_2$ at which $\det[h_i(x_j)\colon 1 \leqslant i, j \leqslant 2]$ is different from 0 at $\lambda_n^+$. Then, the summability of the last display for *both* values of $x$, combined with the fact that $\det B = 1$ on spec $Q$, forces the summability of

$$(\lambda' - \lambda_n^+)^{-1}\,\tfrac{1}{2}(b_{11} + b_{22}) \geqslant (\lambda' - \lambda_n^+)^{-1}(b_{11}b_{22} - b_{12}^2)^{1/2} = (\lambda' - \lambda_n^+)^{-1}$$

and this is contradictory. The presence of an interior pole has been confirmed for most values of $x$. Its motion is easily deduced. Let the location of the root be $\mu_n(x)$. Then, from the vanishing of $h_-(x)\,h_+(x)$ at $\lambda = \mu_n(x)$, one finds[26] $(h_-h_+)' + (h_-h_+)^{\cdot}\,\mu_n' = 0$, whence

$$\mu_n'(x) = -\frac{(h_-h_+)'}{(h_-h_+)^{\cdot}} = \frac{\text{signature}}{(h_-h_+)^{\cdot}}$$

[26] The dot signifies differentiation by $\lambda$.

in which the signature $-(h_- h_+)'$ is $-1$ or $+1$ according as $h_-(x)=0$ or $h_+(x)=0$, in agreement with the convention of Step 2. This shows that the pole moves right or left as dictated by its signature until it hits $\lambda_n^-$ or $\lambda_n^+$, whereupon it changes signature and moves back in the opposite direction, *unless it should just disappear*. But this cannot happen while the pole is inside: in fact, the number $\mu_n'(x)$ is just the residue of $j(x)$ at the pole and does not vanish, by inspection, so disappearance (=the vanishing of the residue) is possible only at an end, in which case the pole immediately reappears moving oppositely, so one may say that *a genuine (or a virtual) pole of $j(x)$ is present for every value of $x$*. Let $\mathfrak{p}_n$ be the pair comprised of the *projection* $\mu_n$ and the *signature* $\pm 1$ previously ascribed if the projection is inside the gap (and indifferently ascribed if it is not). The content of Steps 1–4 is that the family $[\mathfrak{p}_n: n \geqslant 1]$ is a realistic Hill divisor associated with the periodic/antiperiodic spectrum $\lambda_0^+ < \lambda_1^- < \lambda_1^+ < \lambda_2^- < \lambda_2^+ \cdots$, and it is natural to hope that one could recover $Q'$ from it by standard methods.[27] The remaining steps implement this idea.

**Step 5.** The residue of $j(x)$ at its $n$th pole is $\mu_n'(x)$, so[28]

$$j(x) = -\left[\frac{h'_+(x)}{h_+(x)} + \frac{h'_-(x)}{h_-(x)}\right]^{\cdot} = -\sum_{n=1}^{\infty} (\mu_n - \lambda)^{-2} \mu_n'$$

Now, if $\lambda < 0$ and if $df_{00}^x$ is the sum of the old mass distributions $df_\pm^0$ updated for the displacement of the origin from 0 to $x$, then

$$\begin{aligned}\sum_{n=1}^{\infty} (\mu_n - \lambda)^{-2} |\mu_n'| &= \frac{1}{\pi}\int_0^{\infty} (\lambda' - \lambda)^{-2}\, df_{00}^x(\lambda') \\ &= -\left[\frac{h'_-(x)}{h_-(x)} - \frac{h'_+(x)}{h_+(x)}\right]^{\cdot} \\ &= \left[-\frac{1}{G_{xx}(\lambda)}\right]^{\cdot}\end{aligned}$$

and since this is under good control, the first display can be integrated *under* the sum, first with regard to $x$, and second with regard to $\lambda$. This produces

$$\frac{h_-(x)\, h_+(x)}{h_-(0)\, h_+(0)} = \prod_{n=1}^{\infty} \frac{\mu_n(x) - \lambda}{\mu_n(0) - \lambda} \times \text{a function of } x \text{ alone}$$

with good convergence of the product in view of $\lambda_n^- \leqslant \mu_n \leqslant \lambda_n^+$ and the common development $n^2\pi^2 + c_0 + c_1 n^{-2} + \cdots$ of $\lambda_n^-$ and $\lambda_n^+$. The final fac-

[27] McKean and Trubowitz[12] is recommended for background at this juncture.

[28] The fact that $j$ is the difference of positive harmonic functions is used.

tor is found to be unity by noting that both the product and the left-hand side take the value 1 at $\lambda = -\infty$, the upshot being that

$$2h_-(x)\,h_+(x) = D(\lambda) \prod_{n=1}^{\infty} (n\pi)^{-2}[\mu_n(x) - \lambda]$$

with a factor $D$ depending upon $\lambda$, but not upon $x$. This was the aim of the present step.

**Step 6** is to prove that $1/D = (\Delta^2 - 1)^{1/2}$, $\Delta$ being the Hill discriminant for $Q$. It follows that

1. The potential of $Q'$ can be expressed by the customary *trace formula*:

$$\lambda_0^- + [\lambda_1^- + \lambda_1^+ - 2\mu_1(x)] - [\lambda_2^- + \lambda_2^+ - 2\mu_2(x)] + \cdots$$

as can be verified by routine estimation starting from

$$[h_-(x)\,h_+(x)]^{-1} \approx 2[q(x) - \lambda]^{1/2} \qquad (\lambda \downarrow -\infty)$$

2. Moreover,

$$\mu_n'(x) = \frac{\operatorname{sign} \mathfrak{p}_n}{(h_- h_+)^\cdot} = \pm \frac{2(\Delta^2 - 1)^{1/2} \text{ taken at } \mu_n}{(n\pi)^{-2} \prod_{m \neq n} (m\pi)^{-2} (\mu_m - \mu_n)}$$

this being the customary Hill recipe for the motion of the divisor $[\mathfrak{p}_n : n \geqslant 1]$. Condition 2, combined with 1, confirms that *$Q'$ is a Hill operator belonging to the additive class of $Q$.*

*Proof That* $1/D = (\Delta^2 - 1)^{1/2}$. The radical $(\Delta^2 - 1)^{1/2}$ is taken positive for $\lambda < 0 = \lambda_0^+$ and is then determined by continuation off the cut $[0, \infty)$. $D$ is real and analytic in the gaps, by inspection. It is also imaginary on the upper bank of any band (and conjugate imaginary on the lower bank). This follows from[9] $1 = \det B = 1 - |[h_-, h_+^*]|^2$ in the bands, with the implication that $h_+^* = r_{11} h_-$, the factor $r_{11} = [h_+^*, h_+]$ being imaginary, so that $2h_-(x)\,h_+(x)$ is likewise imaginary in the bands. Now $D^2$ is automatically analytic in the slit plane $\mathbb{C}$-spec $Q$, and since it satisfies the reality condition $[D^2(\lambda^*)]^* = D^2(\lambda)$ and is real in the open bands, it is also analytic in the whole complex plane, except perhaps at $\lambda_0^+, \lambda_1^-, \lambda_1^+, \lambda_2^-$, etc. The origin is now adjusted so that $h_-(0)\,h_+(0)$ tends to $-\infty$ at the left end, and to $+\infty$ at the right end of every gap, as in Step 4, the right end of the infinite gap $(-\infty, 0]$ included. Then $\lambda_n^- < \mu_n < \lambda_n^+$ for every $n \geqslant 1$, $D$ is comparable to $2h_-(0)\,h_+(0)$ at each of the possible poles of $D^2$, and the fact that the spectral weight of $Q'$ is without singular part implies that such poles are of degree precisely 1. Now $\Delta^2-1$ is analytic in the whole plane, with (simple) roots at the poles of $D^2$, so the product $F=(\Delta^2-1)^{1/2}D$ is analytic and root-free in the whole plane, and tends to 1 at $-\infty$ in view of the estimates

$$(\Delta^2-1)^{1/2}\approx\exp(-\lambda)^{1/2}$$

$$\prod_{n=1}^{\infty}(n\pi)^{-2}(\mu_n-\lambda)\approx(-\lambda)^{-1/2}\exp(-\lambda)^{1/2}$$

$$m_{11}\approx(-\lambda)^{-1/2}$$

The point at issue is whether the function $F$ is identically 1, and this is now confirmed by checking that it is of *exponential type*: being root-free, it could only be of the form $\exp(A+B\lambda)$ and $A=B=0$ if $F\approx 1$ at $-\infty$. The proof is not difficult: $\lg^+|F[r\exp(\sqrt{-1}\,\theta)]|$ is majorized by $\lg r+\lg^+|m_{11}(r\exp(\sqrt{-1}\,\theta))|$ on nice circles $r=(n+1/2)^2\pi^2\uparrow\infty$, by routine estimation, and, for the rest, it suffices to confirm the finiteness of

$$I=\int_1^{\infty}r^{-2}\,dr\int_0^{2\pi}\lg^+|m_{11}(r\exp(\sqrt{-1}\,\theta))|\,d\theta$$

Now Item 5, Section 2 applies without change to show that $m_{11}(k^2)$ is an outer function in the upper half-plane $[k=a+\sqrt{-1}\,b\colon b>0]$, so that

$$\begin{aligned}\int_0^{2\pi}&\lg^+|m_{11}(r\exp(\sqrt{-1}\,\theta))|\,d\theta\\ &=2\int_0^{\pi}\lg^+|m_{11}|(r\exp(\sqrt{-1}\,\theta))\,d\theta\\ &=2\int_0^{\pi}d\theta\,\frac{r^{1/2}\sin\theta}{\pi}\int_{-\infty}^{\infty}\frac{\lg^+|m_{11}|((k')^2)}{|k'-r^{1/2}\exp(\sqrt{-1}\,\theta)|^2}\,dk'\\ &=\frac{2}{\pi}\int_0^{\infty}\lg^+|m_{11}|\,\frac{dk'}{k'}\lg\left|\frac{k'+r^{1/2}}{k'-r^{1/2}}\right|\end{aligned}$$

But

$$\int_1^{\infty}r^{-2}\,dr\,\lg\left|\frac{k'+r^{1/2}}{k'-r^{1/2}}\right|$$

approximates $4(k')^{-1}$ as $k\uparrow\infty$ and $\frac{4}{3}k'$ as $k'\downarrow 0$, so $I$ is controlled by

$$\int_0^{\infty}\frac{\lg^+|m_{11}|}{(k')^2+1}\,dk'<\infty$$

The proof is finished.

## 3. THE FINITE-DIMENSIONAL ADDITIVE CLASS

The additive class of $Q$ is now assumed to be a (smooth) compact or noncompact manifold of dimension $g < \infty$. This case is coextensive with the Neumann system,[29] as will be seen below.

**Discussion of spec $Q$.** The first task is to prove that spec $Q$ is comprised of consecutive, closed, finite *bands* $[\lambda_{i-1}^+, \lambda_i^-]$ $(i = 1,..., g)$, followed by an infinite band $[\lambda_g^+, \infty)$, and that the (*honest*) bands that do not collapse to singletons (*bound states*) have (a) nonsingular spectral weight and (b) additive invariant $\det B = 1$. This determines the unimodular spectral class of $Q$.

**Step 1.** KDV (3) is devoted to the integration of the vector fields (infinitesimal additions) $X: Q \to 2G'_{xx}(\lambda)$ $(\lambda < 0)$[30] to produce commuting flows in the (suitably closed) additive class of $Q$. This class is of dimension $g$, so that any $g+1$ fields $X_i$ formed for $\lambda'_i < 0$ $(i = 0,..., g)$ have a dependence: $c_0 X_0 + \cdots + c_g X_g = 0$ with $(c_0,..., c_g) \neq 0$. Now, for fixed $\lambda'_0 < 0$,[(10)]

$$X_0 m_{11}(\lambda) = (\lambda - \lambda'_0)^{-1}[m_{11}(\lambda)\, m_{12}(\lambda'_0) - m_{12}(\lambda)\, m_{11}(\lambda'_0)]$$

$$2X_0 m_{12}(\lambda) = (\lambda - \lambda'_0)^{-1}[m_{11}(\lambda)\, m_{22}(\lambda'_0) - m_{22}(\lambda)\, m_{11}(\lambda'_0)] + m_{11}(\lambda)\, m_{11}(\lambda'_0)$$

$$X_0 m_{22}(\lambda) = (\lambda - \lambda'_0)^{-1}[m_{12}(\lambda)\, m_{22}(\lambda'_0) - m_{22}(\lambda)\, m_{12}(\lambda'_0)] + m_{12}(\lambda)\, m_{11}(\lambda'_0)$$

so the vanishing of $c_0 X_0 + \cdots + c_g X_g$ implies an identity for the fundamental matrix:

$$\bar{m}_{11} M = m_{11} \bar{M}$$

with $\bar{M} = [\bar{m}_{ij}: 1 \leqslant i, j \leqslant 2]$ and

$$\bar{m}_{11}(\lambda) = c_0(\lambda - \lambda'_0)^{-1}\, m_{11}(\lambda'_0) + \cdots + c_g(\lambda - \lambda'_g)^{-1}\, m_{11}(\lambda'_g)$$

with $\bar{m}_{12}$ and $\bar{m}_{22}$ being formed in the same way except that $c_0 m_{11}(\lambda'_0) + \cdots + c_g m_{11}(\lambda'_g)$ is added to $\bar{m}_{22}$; in particular, $\bar{m}_{11}$ is a nonvanishing rational function of degree $\leqslant g+1$. Now $\bar{M}$ is real and pole-free on $[0, \infty)$, so $dF = (\bar{m}_{11})^{-1}\, \bar{M} \times df_{11}$ is a $(2 \times 2)$ rational multiple of $df_{11}$ and

$$(\det dF)^{1/2} = (\det B)^{1/2}\, d\lambda = |\bar{m}_{11}|^{-1}[\bar{m}_{11}\bar{m}_{22} - (\bar{m}_{12})^2]^{1/2} \times df_{11}$$

This was the aim of Step 1. The tacit assumption that $\bar{m}_{11}$ is root-free on $[0, \infty)$ is not needed. The issue is raised only at jumps of $df_{11}$ and it is easy to see that $\bar{m}_{12}$ and $\bar{m}_{22}$ vanish at least as hard as $\bar{m}_{11}$ at such a place: in short, the formulas are valid in every case.

[29] See Ref. 13 for background.

[30] $G_{xy}(\lambda)$ is Green's function, as before. The prime signifies differentiation by $x$ *on diagonal.*

**Step 2** is to check that *the singular part of $df_{11}$ is comprised of at most g singletons.*

*Proof.* The rational function $(\bar{m}_{11})^{-2}[\bar{m}_{11}\bar{m}_{22}-(\bar{m}_{12})^2]=-m_{00}$ appearing in the formula for $(\det B)^{1/2}$ vanishes on the singular support of $df_{11}$, so the presence of a singular continuum forces $m_{00}$ to vanish identically, contradicting $m_{11}^2 m_{00}=-\det M=1$. Now let $m_{00}$ vanish at a singleton of $df_{11}$. Then $m_{11}=\pm(m_{00})^{-1/2}$ cannot be balanced nearby if $m_{00}$ has a *simple* root, so the numerator $\det \bar{M}$ of $m_{00}$ has roots of total degree at least $2g+2$ on the cut and an extra root of degree at least 2 at $\infty$. This is too much, since the degree of this function is at most $2g+2$, so it vanishes identically, and that is not the case either: in the vicinity of the pole $\lambda_i'$, $\det \bar{M}\approx c_i^2(\lambda-\lambda_i')^{-2}\det M$, and the vanishing of the left-hand side at every pole forces $c_i=0$ for every $i=0,\ldots,g$.[31]

**Step 3** is to check that $\det B$ *is identically* 1 *on the nonsingular part* spec′ $Q$ *of* spec $Q$.

*Proof.* $m_{11}^2=1/m_{00}$ is a rational function taking *real* values on the cut. Now,[32] $m_{11}=a_{11}+\sqrt{-1}\,b_{11}$, and the reality of $m_{11}^2$ implies that $a_{11}b_{11}$ vanishes; similarly, $m_{22}^2=(m_{11}\bar{m}_{22}/\bar{m}_{11})^2$ is real, so $a_{22}b_{22}$ vanishes, too, and likewise $a_{12}b_{12}$. Now $-1=\det M$ is used to confirm that $\det B=\det A+1$. This reduces to $\det B=1-a_{12}^2$ on spec′ $Q$, since *sp dF* is now proportional to $b_{11}\,d\lambda$ with the implication that $b_{11}>0$ and $a_{11}=0$ at almost every point of spec′ $Q$. It remains to prove that $a_{12}=0$ at almost every point of spec′ $Q$. But $a_{12}\neq 0$ implies $b_{12}=0$, and this occurrence on a subset of spec′ $Q$ of positive measure violates $b_{11}=b_{12}\bar{m}_{11}/\bar{m}_{12}$ *unless* the rational function $\bar{m}_{12}$ vanishes identically, in which case $m_{12}$ itself vanishes identically, with the implication[(9)] that $Q$ is symmetrical about $x=0$. Now the desired conclusion ($\det B=1$) is insensitive to displacement of the origin, so it can fail only if $Q$ is symmetrical about *every* origin. This happens only if $Q=-D^2$, up to an additive constant, in which case $\det B=1$ anyhow. This exceptional case ($g=0$) is left aside below.

**Step 4** is to note that spec $Q$ *consists of a finite number of consecutive disjoint (short) bands, followed by a (long) band extending to* $\infty$. The short bands may be *honest* or they may collapse to points (*bound states*); it is on the latter that the singular part of the spectral weight resides. The proof starts by noting that

$$-m_{11}^2=(\bar{m}_{11})^2(\det \bar{M})^{-1}=b_{11}^2-a_{11}^2-2\sqrt{-1}\,a_{11}b_{11}=b_{11}^2-a_{11}^2$$

[31] $\det M=-1$ is used again.
[32] $M=A+\sqrt{-1}\,B=[a_{ij}+\sqrt{-1}\,b_{ij}\colon 1\leqslant i,j\leqslant 2]$.

is (a) rational, (b) real, (c) positive at most points of spec $Q$ $[b_{11}>0=a_{11}]$, (d) negative outside $[b_{11}=0]$, and (e) ultimately positive [spec $Q$ extends to $\infty$]. It follows that the *nonsingular* part of spec $Q$ consists of a finite number of (honest) short bands followed by a long band extending to $\infty$. The rest of the proof depends upon the fact that, *in any honest closed band, short or long, $b_{11}^2$ coincides with a rational function, free of interior roots and poles, having, at the ends of the band, poles of degree at most* 1.

*Proof.* $b_{11}^2=-m_{11}^2$ in the band and any interior root or pole is of degree at least 2, $b_{11}^2$ being of one sign. But such poles are excluded by the summability of $b_{11}$ in the band; similarly, such roots are excluded by the summability of $\mathrm{Im}(-1/m_{11})=1/b_{11}$. Now, $b_{11}^2$ takes both signs in the vicinity of an endpoint, so the argument fails at such a place: it shows only that the pole (or root) is of degree 1 or less.

The disposition of the *singular* spectral weight can now be clarified: *it is carried by collapsed bands (bound states) disjoint from the honest bands.*

*Proof.* The existence of a lump of singular spectral mass at a point $a$ of a closed honest band implies that $\mathrm{Im}\, m_{11}(a+\sqrt{-1}\, b)$ is underestimated by a multiple of $1/b$ as $b\downarrow 0$. But then the rational function $m_{11}^2$ has a pole of degree 2 or more, and that was ruled out just now.

**Step 5** is to determine the form of $m_{11}$. Let $[\lambda_{i-1}^+, \lambda_i^-]$ $(i=1,\ldots,n)$ be the (honest and dishonest) short bands, let $[\lambda_n^+, \infty)$ be the long band, and introduce the *radical*:

$$R(\lambda)=(\lambda_0^+-\lambda)^{1/2}\prod_{i=1}^{n}[(\lambda_i^--\lambda)(\lambda_i^+-\lambda)]^{1/2}$$

It is to be proved that $m_{11}=P/R$ *in which $P$ is a polynomial of degree $n$ having (mostly) one simple root in each of the closed gaps* $[\lambda_i^-, \lambda_i^+]$ $(i=1,\ldots,n)$ *separating the bands.*

*Proof.* $m_{11}$ is imaginary on the bands and real on the gaps, as is $R$, so $P=Rm_{11}$ is real on the cut and so extends to a meromorphic function of $\lambda\in\mathbb{C}$, with possible poles at the bound states and at the ends of the honest bands. The fact is that no such poles are present: $R$ knocks out any pole of the first kind, and at the ends of an honest band it cuts the degree of any pole down to $\leqslant 1/2$ and so to 0. It follows that $P$ is entire and, since its square is rational, it must be a polynomial, of degree $n$ in view of the estimate $P=Rm_{11}\approx(-\lambda)^{n+1/2-1/2}$ at $-\infty$. Besides, $P$ has roots only in the closed gaps, since it cannot vanish off the cut or in the interior of an honest band where $m_{11}=\sqrt{-1}\, b_{11}\neq 0$: in fact, it has (mostly) just one (simple) root *inside* each gap, since a small displacement of the origin ensures this outcome, just as in Step 5, Section 2: in short, the roots $\mu_i$ of $P$ fall in the closed gaps $[\lambda_i^-, \lambda_i^-]$ $(i=1,\dots,n)$ and $m_{11}=P/R$.

**Step 6** introduces the *divisor* $\mathfrak{p}_1,\dots,\mathfrak{p}_n$ comprising the *roots* $\mu_i$ $(i=1,\dots,n)$ of $P(\lambda)=0$ and the associated *signatures*, as in Section 2. The roots move (in response to displacement of the origin) back and forth in their gaps according to the rules of Section 2: in fact, nothing is changed except that the only way for roots to meet bound states is to collide in pairs, one root coming from the left and one from the right; for details, see the amplification below.

**Step 7** is to check that *$Q$ is determined by the divisor $[\mathfrak{p}_i: i\leqslant n]$ provided the latter is in general position, meaning that each root is inside its gap.* It is a by-product that *the dimension $g$ of the additive class cannot exceed the number $n$ of short (honest and collapsed) bands.*

*Proof.* The divisor determines $m_{11}$ by the formula of Step 5: $m_{11}=P/R$. Now consider the functions $h'_+(0)/h_+(0)$ and $-h'_-(0)/h_-(0)$ determining the side operators $Q_-^0$ and $Q_+^0$. Their sum is $-1/m_{11}$, which is in hand, while their difference is the rational function $2m_{12}/m_{11}=2\bar{m}_{12}/\bar{m}_{11}$. The latter has (simple) poles at the roots $\mu_i$ $(i\leqslant n)$ of $m_{11}$ and no others, as one can easily check. The corresponding residues

$$r_i=2m_{12}(\mu_i)/m^{\cdot}_{11}(\mu_i)= -2\ \mathrm{sign}\ \mathfrak{p}_i/m^{\cdot}_{11}(\mu_i) \qquad (i\leqslant n)$$

are also determined by the divisor, and the function itself vanishes at $\infty$ in view of the fact that $h'_+(0)/h_+(0)$ and $-h'_-(0)/h_-(0)$ have the common development $[q(0)-\lambda]^{1/2}+O(\lambda^{-3/2})$ at $-\infty$: in short, the divisor determines $-1/m_{11}$, $2m_{12}/m_{11}$, and so both $h'_+(0)/h_+(0)$ and $-h'_-(0)/h_-(0)$, individually.

**Step 8** is to pin down the dimension $g=n$.

*Proof.* The fields $X_0: Q\to 2G'_{xx}(\lambda'_0)$ $(\lambda'_0<0)$ are employed. By the first formula of Step 1,

$$0=(X_0 m_{11})(\mu_i)+m^{\cdot}_{11}(\mu_i)\,X_0\mu_i= -\frac{m_{11}(\lambda'_0)}{\mu_i-\lambda'_0}\,m_{12}(\mu_i)+m^{\cdot}_{11}(\mu_i)\,X_0\mu_i$$

so that[33]

$$X_0\mu_i=\frac{m_{11}(\lambda'_0)\ \mathrm{sign}\ \mathfrak{p}_i}{(\lambda'_0-\mu_i)\,m^{\cdot}_{11}(\mu_i)} \qquad (i\leqslant n)$$

[33] $-\mathrm{sign}\ \mathfrak{p}_i=m_{12}(\mu_i)$.

Now let $X_1,\dots, X_n$ be the fields corresponding to any fixed $\lambda_1' < \cdots < \lambda_n' < 0$ and compute

$$\det[X_i\mu_j\colon 1 \leqslant i, j \leqslant n] = \det \frac{m_{11}(\lambda_i') \operatorname{sign} \mathfrak{p}_j}{(\lambda_i' - \mu_j)\, m_{11}^{\cdot}(\mu_j)}$$

This cannot vanish, by inspection, so the infinitesimal additions allow us to move the divisor at pleasure (in the small): in short, the additive class has at least $n$ ($\leqslant g$) degrees of freedom. The equality $n = g$ is now confirmed by Step 7. This finishes the discussion of spec $Q$.

**Additive and Unimodular Classes.** The unimodular class of $Q$ is determined by the preceding spectral discussion: it consists of all operators $Q'$ having the same (honest and collapsed) spectral bands as $Q$, with singular weight confined to the bound states, nonsingular weight confined to the honest bands, and det $B = 1$ on the latter. Now it is a triviality to confirm by familiar methods that, after a small displacement of the origin, such an operator $Q'$ determines, and is determined by, a divisor in general position, just as for $Q$. Then the identification of the additive and the unimodular classes comes down to this: *every divisor in general position appears already in the additive class.* The proof employs the infinitesimal additions of Step 8 to move the roots of $m_{11}$ to any (general) position one pleases, whereupon it suffices to prove that the *signatures* of the general divisor can be changed at will *within the additive class.*

*Proof.*[34] Let $\mathfrak{p} = (\lambda, \pm 1)$ be fixed with $\lambda < 0$ and let $e(x, \mathfrak{p})$ be $h_-(x)$ or $h_+(x)$ according to the signature of $\mathfrak{p}$. The addition $A^{\mathfrak{p}}\colon Q \to Q - 2D^2 \lg e(x, \mathfrak{p})$ can be repeated, with the following result:

$$A^{2\mathfrak{p}}Q = Q - 2D^2 \lg \int_x^\infty h_+^2 \qquad \text{if} \quad \operatorname{sign} \mathfrak{p} = +1$$

$$= Q - 2D^2 \lg \int_{-\infty}^x h_-^2 \qquad \text{if} \quad \operatorname{sign} \mathfrak{p} = -1$$

as can be readily confirmed by the rule for composite addition:

$$A^{\mathfrak{p}'}A^{\mathfrak{p}}Q = Q - 2D^2 \lg[e(x, \mathfrak{p}'), e(x, \mathfrak{p})]$$

by taking $\mathfrak{p}' = (\lambda + \Delta\lambda, \pm 1)$, inserting $(\Delta\lambda)^{-1}$ after the logarithm, and making $\Delta\lambda \downarrow 0$. One applies $A^{2\mathfrak{p}}$ when $\mathfrak{p} = \mathfrak{p}_0 = (\mu_0, \pm 1)$ is a point of the divisor of $Q$; this is not a *true addition* as construed before, $\mu_0$ not being to the left of spec $Q$. It is to be proved that *$Q' = A^{2\mathfrak{p}}Q$ belongs to the additive*

[34] McKean and van Moerbeke,[11] p. 221, served as a model.

*class of $Q$ and has the same divisor as $Q$ except that the sign of* $\mathfrak{p}_0$ *is reversed.* Let $e_0 = e(\cdot, \mathfrak{p}_0)$ for brevity and sign $\mathfrak{p}_0 = +1$ for definiteness, let $Z = \int_x^\infty e_0^2$, and let $A = A^{2\mathfrak{p}_0}$.

**Item 1:**

$$Ah_+ = h_+ - (\lambda - \mu_0)^{-1} Z^{-1} e_0 [e_0, h_+]$$
$$= h_+ + Z^{-1} e_0 \int_x^\infty e_0 h_-$$

*off the cut* $[0, \infty)$, i.e., this function solves $Q'h = \lambda h$ and is of class $L^2[0, \infty)$.

*Proof.* $Q'h = \lambda h$, by routine computation, and $\int_0^\infty |Ah_+|^2$ cannot diverge, since

$$I = \int_0^n \left[ Z^{-1} e_0 \int_x^\infty e_0 h_+ \right]^2 = \int_0^n \left[ \int_x^\infty e_0 h_+ \right]^2 dZ^{-1}$$
$$= Z^{-1} \left[ \int_x^\infty e_0 h_+ \right]^2 \Bigg|_0^n + 2 \int_0^n Z^{-1} e_0 h_+ \int_x^\infty e_0 h_+$$
$$\leqslant \int_n^\infty h_+^2 - I^{1/2} \left[ \int_0^\infty h_+^2 \right]^{1/2}$$

for $n \uparrow \infty$.

**Item 2.** This item is similar:

$$Ah_- = h_- - (\lambda - \mu_0)^{-1} Z^{-1} e_0 [e_0, h_-] = h_- - Z^{-1} e_0 \int_{-\infty}^x e_0 h_-$$

also, $[Ah_-, Ah_+] = 1$ is inherited from $[h_-, h_+] = 1$.

**Item 3.** *$Q'$ belongs to the unimodular spectral class of $Q$*: indeed, $A\, dF = G\, dF\, G^\dagger$ with the unimodular factor

$$G = \begin{bmatrix} 1 & 0 \\ c(\lambda - \mu_0)^{-1} & 1 \end{bmatrix}, \qquad c = (-1) \times \text{the reciprocal of } e^{\cdot}(0, \mathfrak{p}_0)$$

by routine computation from Items 1 and 2.

**Item 4.** $Q'$ is determined by a divisor $\mathfrak{p}_i'$ $(i \leqslant g)$ in view of Item 3. I compute it. $Ah_+$ vanishes identically at $\lambda = \mu_0$: in fact, at this place, $e_0 = h_+$, up to a factor that may be taken as unity, $Ah_+ = h_+ -$

$Z^{-1}h_+[h_+, h^\cdot_+]$, and $[h_+, h^\cdot_+] = \int_x^\infty h_+^2 = Z$; similarly, $Ah_-$ is infinite. What one must do is to divide $Ah_+$ by $\lambda - \mu_0$ and to multiply $Ah_-$ by the same factor. Then $Ah_+(0) = (\lambda - \mu_0)^{-1}\, h_+(0)$ and $Ah_-(0) = (\lambda - \mu_0)\, h_-(0)$, of which the first vanishes at the projection of $\mathfrak{p}_i$ if sign $\mathfrak{p}_i = +1$ *with the sole exception of* $\mathfrak{p}_0$, while the second vanishes at the projection of $\mathfrak{p}_i$ if sign $\mathfrak{p}_i = -1$, *and at the projection of* $\mathfrak{p}_0$ *as well*: in short, *the divisor* $\mathfrak{p}_i'$ $(i \leqslant g)$ *of* $Q'$ *is the same as that of* $Q$ *except that the signature of* $\mathfrak{p}_0$ *is reversed.*

**Item 5.** *$Q'$ belongs to the (suitably closed) additive class of $Q$.* This will complete the proof that the additive and the unimodular spectral classes are the same.

*Proof.* The field $X\colon Q \to 2G'_{xx}(\lambda)$ $(\lambda < 0)$ is expressed as $m'_{11}$ with $m_{11} = P/R$, as in Step 5, and everything *updated* by displacement of the origin from 0 to $x$. Fix $\lambda_0' < \cdots \lambda_g' < 0$ and let $X_0, \ldots, X_g$ be the corresponding fields. Then

$$X = \frac{1}{R(\lambda)} \sum_{j=0}^{g} \prod_{i \neq j} \frac{\lambda - \lambda_i'}{\lambda_j' - \lambda_i'} R(\lambda_j')\, X_j$$

by interpolation of $P(\lambda)$ off $\lambda_0', \ldots, \lambda_g'$, and the fact that $X$ is infinitesimal addition[35] permits one to write

$$A^{\mathfrak{p}} = A^{\mathfrak{o}} \exp\left( -\int_{\mathfrak{o}}^{\mathfrak{p}} X\, d\lambda \right) = A^{\mathfrak{o}} \exp\left( -\sum_{j=0}^{g} \int_{\mathfrak{o}}^{\mathfrak{p}} \omega_j X_j \right)$$

in which $\mathfrak{o}$ is the point $(-1, -1)$ and

$$\omega_j = \prod_{i \neq j} \frac{\lambda - \lambda_i'}{\lambda_j' - \lambda_i'} R(\lambda_j') \frac{d\lambda}{R(\lambda)}$$

[35] $A^{\mathfrak{p}'} A^{-\mathfrak{p}} = I - \Delta\lambda X + \cdots$ for $\mathfrak{p} = (\lambda, +1)$ and $\mathfrak{p}' = (\lambda + \Delta\lambda, +1)$; see KDV (1).

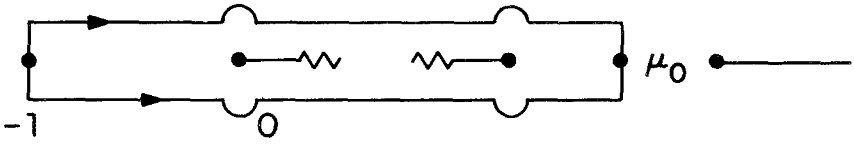


Figure 2

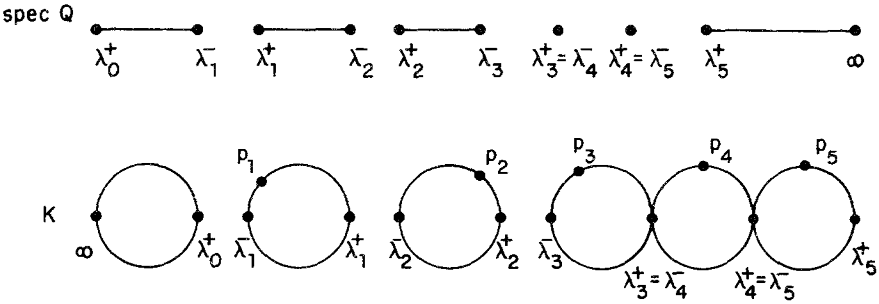


Figure 3

is a differential of the second kind with simple pole at $\infty$. It follows that

$$A = A^{2\mathfrak{p}_0} = A^{2\mathfrak{o}} \exp\left[ -\sum_{j=0}^{g} \int_C \omega_j X_j \right]$$

$$= A^{2\mathfrak{o}} \exp\left[ -2 \sum_{j=0}^{g} \int_{[-1,\mu_0]-\text{spec}\,Q} \omega_j X_j \right]$$

in which $C$ is the sum of the paths indicated in Fig. 2 and the final form of the exponentiated integral is due to cancellation of radicals on the bands of spec $Q$, leaving a *real* combination of $X_0,..., X_g$. The upshot is that $A$ can be approximated by (infinitesimal) additions, whence $AQ = Q'$ belongs to the additive class.

**Amplification.** I clarify the motion of the divisor $\mathfrak{p}_i$ $(i \leqslant g)$ in response to displacement of the origin. The point $\mathfrak{p}_i$ depends upon $x$, its projection $\mu_i$ moving back and forth in its gap according to the rule $\mu_i' =$ sign $\mathfrak{p}_i/m_{11}^{\cdot}(\mu_i)$, in which $m_{11}$ is the updated function $2h_-(x)\, h_-(x)$. The turning around of, e.g., $\mu_2$ at the bottom of the honest band $[\lambda_2^+, \lambda_3^-]$ is easy to understand: to the left of $\lambda_2^+$, $m_{11}(\lambda)$ is a nonvanishing multiple of $(\mu_2 - \lambda)(\lambda_2^+ - \lambda)^{-1/2}$, so that, with sign $\mathfrak{p}_2 = +1$, $\mu_2'$ is (approximately) proportional to $(\lambda_2^+ - \mu_2)^{1/2}$, and the substitution $\mu_2 = \lambda_2^+ - \sin^2\theta$ shows that $\theta$ moves at (nearly) constant speed through the collision of $\mu_2$ with $\lambda_2^+$. The collision of roots at bound states is different. Let $\lambda_2^+ = \lambda_3^-$ be such a bound state and let $\mu_2$ hit it at $x = 0$. Then for small $x < 0$, $\mu_2 < \lambda_2^+$ is traveling to the right, $\mu_3$ is to the right of $\lambda_3^-$, and $m_{11}(\lambda)$ has a root at $\mu_2$,

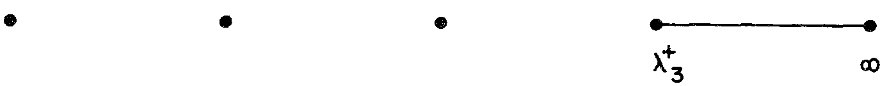


Figure 4

822/46/5-6-23


Figure 5

a pole at $\lambda_2^+$, and a root at $\mu_3$. Now the total degree of $m_{11}$ in the vicinity of $\lambda_2^+$ cannot change in the small, so, *at collision*, $m_{11}$ has lost its pole and $\lambda_2^+$ has joined the spectrum of $Q_+^0$. But $\lambda_2^+ = \lambda_3^-$ is a *bound state*, so $\lambda_2^+ \in$ spec $Q_+^0$ means that the corresponding eigenfunction $e_2$ *vanishes at* $x=0$, i.e., $\lambda_2^+ = \lambda_3^-$ belongs to spec $Q_-^0$ as well, which is to say that $\mu_3$ has arrived at $\lambda_3^-$, bringing an extra root to $m_{11}$, for a total count of two roots + one pole = one (simple) root at $\lambda_2^+ = \lambda_3^-$. To sum up: the roots $\mu_2$ and $\mu_3$ collide at $\lambda_2^+ = \lambda_3^-$ precisely at the ($\geqslant 2$) roots of $e_2(x) = 0$. The rule of collisions at other bound states is similar. The only exception concerns $\mu_1$, which cannot hit $\lambda_1^-$ if the latter is a bound state, in view of the fact that a ground state eigenfunction is root-free. Now the motion of the divisor can be integrated in a simple way. The radical $R$ is regarded as defining a hyperelliptic curve $K$, as in Fig. 3, with double points coming from the bound states. This suggests the introduction of differentials of the first kind of the form $\omega_j = \lambda^{j-1}\, d\lambda/R(\lambda)$ ($j \leqslant g$), permitting one to map the divisor over to the Jacobi variety $J$ of $K$ where it moves in a straight line at constant speed. It is a by-product that *every possible divisor* $\mathfrak{p}_i$ ($i \leqslant g$) *with projections in the closed gaps actually occurs in the additive class, permitting one to identify that class with the* (*real*) *Jacobi variety.* I do not give details, but compare McKean.[8]

**General Picture.** The upshot of the whole discussion is that the additive class of finite dimension is nothing but the general leaf of the Neumann system of $g$ uncoupled oscillators $x_i^{\cdot\cdot} + \omega_i^2 x_i = 0$ ($i \leqslant g$) with distinct frequencies $\omega_1,\ldots,\omega_g$ constrained to move on the unit sphere $x_1^2 + \cdots + x_g^2 = 1$ by the imposition of the normal force $f = [2(\omega_1^2 x_1^2 + \cdots + \omega_g^2 x_g^2) + I]\, x$; see Refs. 7, 8, and 13 for background. The special case of $g$ solitons appears when all the short bands collapse, as in Fig. 4, while the case of Hill's equation with finitely many gaps appears when all the short bands are honest, as in Fig. 5. The general leaf is obtained from the Hill's leaf by collapse of one or more bands.

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